A Brief Summary of Laurent Series

Let f be a holomorphic function on an annulus $D = \{ z \in \mathbb{C} : r < |z - z_0| < R \}$. Then f can be expanded into a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n},$$

where both series converge on D and uniformly on every subannulus of D concentric with D. Let C be any simple closed contour in D ~~concentric with~~ containing z_0 and oriented once in the counterclockwise direction.

Then
$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta, \quad n = 0, \pm 1, \pm 2, \ldots.$$

We call $\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$ the Laurent series of f at z_0.

<u>Theorem</u>: Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $\sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$ be series such that

- $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges on $\{ z \in \mathbb{C} : |z - z_0| < R \}$,
- $\sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$ converges on $\{ z \in \mathbb{C} : |z - z_0| > r \}$,
- $r < R$.

Then there exists a unique holomorphic function f on $D = \{ z \in \mathbb{C} : r < |z - z_0| < R \}$ such that the Laurent series of f on D is

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}, \quad r < |z - z_0| < R.$$

Proof: Assume $z_0 = 0$. Let $\zeta = \frac{1}{z}$. Then

$$\sum_{n=1}^{\infty} a_{-n}\zeta^n$$

converges on $\{\zeta \in \mathbb{C} : |\zeta| < \frac{1}{r}\}$. Let

$$H(\zeta) = \sum_{n=1}^{\infty} a_{-n}\zeta^n, \quad |\zeta| < \frac{1}{r}.$$

Then H is holomorphic on $\{\zeta \in \mathbb{C} : |\zeta| < \frac{1}{r}\}$. So the function $h(z) = H\left(\frac{1}{z}\right)$ is holomorphic on $|z| > r\}$.

$$\text{and } h(z) = \sum_{n=1}^{\infty} a_{-n} z^{-n}, \quad |z| > r.$$

Now the function $g(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < R$, is holomorphic. So the function f given by

$$f(z) = g(z) + h(z)$$

is holomorphic on $D = \{z \in \mathbb{C} : r < |z| < R\}$. So $\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} z^{-n}$ is the Laurent series of f in D at $z_0 = 0$. To do this, let C be a simple closed contour in D enclosing 0 and oriented once in the counterclockwise direction. Then for all $j = 0, \pm 1, \pm 2, \ldots$,

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z^{j+1}}\,dz = \frac{1}{2\pi i}\int_C \sum_{n=0}^{\infty} a_n z^{n-j-1}\,dz$$

$$= \frac{1}{2\pi i}\int_C \sum_{n=-\infty}^{\infty} a_n z^{n-j-1}\,dz$$

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$$= \frac{1}{2\pi i} a_j \, 2\pi i = a_j.$$

Example: Find the Laurent series of
$$f(z)=\frac{z^2-2z+3}{z-2}$$
on $\{z\in\mathbb{C} : |z-1|>1\}$.

Solution: $z_0=1$, $r=1$, $R=\infty$. $\therefore$ f is holomorphic on $\{z\in\mathbb{C} : 1<|z-1|<\infty\}$.

Now $$\frac{1}{z-2}=\frac{1}{z-1-1}=\frac{1}{z-1}\,\frac{1}{1-\frac{1}{z-1}}$$
$$=\frac{1}{z-1}\sum_{n=0}^{\infty}\frac{1}{(z-1)^n}=\sum_{n=0}^{\infty}\frac{1}{(z-1)^{n+1}}.$$

But
$$z^2-2z+3=z^2-2z+1+2=(z-1)^2+2.$$

$\therefore$ $f(z)$
$$=\left\{(z-1)+1+\frac{1}{z-1}+\frac{1}{(z-1)^2}+\cdots\right\}+\left\{\frac{2}{z-1}+\frac{2}{(z-1)^2}+\cdots\right\}$$
$$=(z-1)+1+\sum_{n=1}^{\infty}\frac{3}{(z-1)^n},\quad |z-1|>1.$$

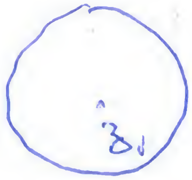
Example Find the Laurent series of $e^{\frac{1}{z}}$ at $z=0$.

Solution $$e^{\frac{1}{z}}=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{z^n}$$
$$=1+\sum_{n=1}^{\infty}\frac{1}{n!}z^{-n},\quad z\neq 0.$$

Why Laurent Series?

Isolated Singularities

Definition: Let $w = f(z)$ be a complex-valued function.
Let $z_0 \in \mathbb{C}$ be $\ni \begin{cases} f \text{ is not holomorphic at } z_0 \\ f \text{ is holomorphic on some punctured} \\ \text{disk of } z_0. \end{cases}$



Then we say that z_0 is an isolated singularity of f.

Let z_0 be an isolated singularity of f. Then f is holomorphic on a punctured disk
$\{ z \in \mathbb{C} : 0 < |z - z_0| < R \}$.

Three Possibilities on $\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$

- If $a_{-n} = 0$ for $n = 1, 2, \ldots$, then we call z_0 a removable singularity of f.
- If $a_{-m} \neq 0$ for some $m \in \mathbb{N}$ and $a_{-n} = 0$ for all $n > m$, then we call z_0 a pole of order m of f.
  A pole of order 1 is called a simple pole.
- If $a_{-n} \neq 0$ for infinitely many $n \in \mathbb{N}$, then we call z_0 an essential singularity of f.

Example: Classify the isolated singularities of $f(z) = e^{\frac{1}{z}}$, $z \neq 0$

Solution 0 is the only isolated singularity of f.

Also,
$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots \quad z \neq 0$$
the Laurent series of $e^{\frac{1}{z}}$

Now note that $a_{-n} = \frac{1}{n!} \neq 0$ for all $n \in \mathbb{N}$.